NASA TT F-9290

INVESTIGATION OF THREE-DIMENSIONAL PHOTOELASTIC MODELS

Kh. K. Aben

Translation of "K issledovaniyu ob" yemnykh fotouprugikh modeley."
Izvestiya Akademii Nauk SSSR,
Otdeleniye Tekhnicheskikh Nauk, Mekhanika i Mashinostroyeniye,
No. 4, pp. 40-46, 1964.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON MARCH 1965

INVESTIGATION OF THREE-DIMENSIONAL PHOTOELASTIC MODELS

/40

ABSTRACT

19509

General equations, somewhat simpler than those found in the literature (refs. 1 and 2), are derived for the photoelastic method. The transformation is made to a rotating coordinate system whose axes are aligned with the quasi-principal stresses. The equivalence of the resultant equations to the Neumann equations is demonstrated. The new equations are used to investigate certain special problems. General relations are derived for the method of characteristic directions, and the application of this method for the investigation of three-dimensional models is considered. Author

1. The propagation of light in an anisotropic inhomogeneous medium is described by equations derived by V. L. Ginzburg (refs. 3 and 4):

$$\frac{d^2E_1}{dz^2} + \frac{\omega^2}{c^2} D_1 = 0, \qquad \frac{d^2E_2}{dz^2} + \frac{\omega^2}{c^2} D_2 = 0 \tag{1.1}$$

where E_i, D_i are the components of the electric field and electric induction vectors, ω is the circular frequency, c is the velocity of light in vacuum, z is measured in the direction of propagation. Bearing in mind the relations

$$D_i = \sum_{j=1}^{2} \varepsilon_{ij} E_j \qquad (i = 1, 2) \tag{1.2}$$

where ϵ_{ij} is the dielectric tensor, we obtain the system of equations

$$-\frac{d^2E_1}{dz^2} = \frac{\omega^2}{c^2}(\varepsilon_{11}E_1 + \varepsilon_{12}E_2), \quad -\frac{d^2E_2}{dz^2} = \frac{\omega^2}{c^2}(\varepsilon_{21}E_1 + \varepsilon_{22}E_2) \tag{1.3}$$

Numbers in the margin indicate pagination in the original foreign text.

In the case of isotropic, optically sensitive materials subjected to elastic deformations, the dielectric tensor ϵ_{ij} is related to the stress tensor σ_{ij} by the expression

$$\varepsilon_{ij} = n^2\delta_{ij} + 2C_0 n\sigma_{ij} + 2C_1 n \sum_{k=1}^{3} \sigma_{kk}\delta_{ij} \tag{1.4}$$

where n is the refractive index of the unstressed material, δ_{ij} is the tensor Kronecker delta, C_0, C_1 are the optical constants.

We will seek a solution of the system (1.3) in the form

$$E_j = A_j e^{-ikz} \quad \left(k = \frac{\omega n}{c},\ j = 1, 2\right) \tag{1.5}$$

The transformation (1.5) takes us from analysis of the functions E_1 and E_2, which are rapidly varying in z, to analysis of the slowly varying functions A_1 and A_2, retaining the same amplitude ratio and phase difference of the components as before.

We have from the relations (1.3) to (1.5)

$$\begin{aligned} &\frac{d^2A_1}{dz^2} - 2ik\frac{dA_1}{dz} + \frac{2k^2}{n}\left(C_0\sigma_{11} + C_1 \sum_{i,j=1}^{3} \sigma_{ij}\delta_{ij}\right)A_1 + \frac{2k^2}{n}C_0\sigma_{12}A_2 = 0 \\ &\frac{d^2A_2}{dz^2} - 2ik\frac{dA_2}{dz} + \frac{2k^2}{n}C_0\sigma_{21}A_1 + \frac{2k^2}{n}\left(C_0\sigma_{22} + C_1 \sum_{i,j=1}^{3} \sigma_{ij}\delta_{ij}\right)A_2 = 0 \end{aligned} \tag{1.6}$$

If the medium is optically isotropic, A_j = const. The optical anisot- /41
ropy of photoelastic models is very slight, hence it may be assumed that dA_j/dz and d^2A_j/dz^2 are of roughly the same order of magnitude.

In view of the fact that the quantity k is large, of the order 10^5, while the constants C_0 and C_1 are small, of the order 10^{-5} or 10^{-6}, the influence of the second derivatives in the system (1.6) on the solution is insignificant. This means that, instead of the system (1.6), we are entitled to investigate the system

$$\frac{dA_1}{dz} = -i\,(C\sigma_{11} + C^*\sigma^1)\,A_1 - iC\sigma_{12}A_2, \quad \frac{dA_2}{dz} = -iC\sigma_{21}A_1 - i\,(C\sigma_{22} + C^*\sigma^1)\,A_2$$

$$\left(C = \frac{C_0k}{n}, \quad C^* = \frac{C_1k}{n}, \quad \sigma^1 = \sum_{i,\,j=1}^{3} \sigma_{ij}\delta_{ij}\right) \tag{1.7}$$

For further simplification of the equations, we go over to the new variables (ref. 5)

$$A_j = B_je^{-i\mu} \qquad \left(\mu = \int C^*\sigma^1\,dz\right) \tag{1.8}$$

This transformation, like the transformation (1.5), does not alter the amplitude ratio or phase difference of the electric vector components.

Taking (1.8) into account, we obtain from (1.7)

$$\frac{dB_1}{dz} = -iC\sigma_{11}B_1 - iC\sigma_{12}B_2, \qquad \frac{dB_2}{dz} = -iC\sigma_{21}B_1 - iC\sigma_{22}B_2 \tag{1.9}$$

Equations (1.9) are general equations of the photoelastic method when the electromagnetic mode is described in stationary coordinates.

2. Let the quasi-principal directions form an angle $\phi(z)$ with the stationary coordinates. Designating the components of the electric vector along the quasi-principal directions by B_j', we have

$$B_1 = B_1'\cos\varphi - B_2'\sin\varphi, \qquad B_2 = B_1'\sin\varphi + B_2'\cos\varphi \tag{2.1}$$

Substituting equations (2.1) into (1.9), we obtain

$$\frac{dB_1'}{dz}\cos\varphi - B_1'\frac{d\varphi}{dz}\sin\varphi - \frac{dB_2'}{dz}\sin\varphi - B_2'\frac{d\varphi}{dz}\cos\varphi +$$
$$+ iC\sigma_{11}(B_1'\cos\varphi - B_2'\sin\varphi) + iC\sigma_{12}(B_1'\sin\varphi + B_2'\cos\varphi) = 0 \tag{2.2}$$

$$\frac{dB_1'}{dz}\sin\varphi + B_1'\frac{d\varphi}{dz}\cos\varphi + \frac{dB_2'}{dz}\cos\varphi - B_2'\frac{d\varphi}{dz}\sin\varphi +$$
$$+ iC\sigma_{21}(B_1'\cos\varphi - B_2'\sin\varphi) + iC\sigma_{22}(B_1'\sin\varphi + B_2'\cos\varphi) = 0 \tag{2.2}$$

We multiply the first equation by sin ϕ, the second by cos ϕ , and subtract; we then multiply the first equation by cos ϕ , the second by sin ϕ , and add. Denoting the quasi-principal stresses by σ_j, we have

$$\frac{dB_1'}{dz} = -iC\sigma_1 B_1' + \frac{d\varphi}{dz} B_2', \qquad \frac{dB_2'}{dz} = -\frac{d\varphi}{dz} B_1' - iC\sigma_2 B_2'. \quad (2.3) \tag{2.3}$$

Equations (2.3) are similar to the equations in partial derivatives of Mindlin and Goodman (ref. 6) but differ appreciably from the most widely recognized photoelastic equations derived by O'Rourke (ref. 7) and Proshko (refs. 8 and 9). Inasmuch as equations (2.3) are simpler than the equations derived by other authors, a question arises as to whether equations (2.3) adequately describe the photoelastic effects in complex-stress models. To answer this, we will demonstrate the equivalence of equations (2.3) to the Neumann equations (ref. 10), which may be regarded as the classical approximation in the theory of photoelasticity.

3. Consider a solution of the system (2.3) in for form /42

$$B_j' = V_j e^{-i\Delta_j} \tag{3.1}$$

where V_j and Δ_j are functions of z.

Substituting equations (3.1) into the system (2.3) and setting the real and imaginary parts of the resultant equations separately equal to zero, we have

$$\begin{aligned}
&\frac{dV}{dz}\cos\Delta_1 - \frac{d\Delta_1}{dz} V_1 \sin\Delta_1 - \frac{d\varphi}{dz} V_2 \cos\Delta_2 + C\sigma_1 V_1 \sin\Delta_1 = 0 \\
-&\frac{dV_1}{dz}\sin\Delta_1 - \frac{d\Delta_1}{dz} V_1 \cos\Delta_1 + \frac{d\varphi}{dz} V_2 \sin\Delta_2 + C\sigma_1 V_1 \cos\Delta_1 = 0 \\
&\frac{dV_2}{dz}\cos\Delta_2 - \frac{d\Delta_2}{dz} V_2 \sin\Delta_2 + \frac{d\varphi}{dz} V_1 \cos\Delta_1 + C\sigma_2 V_2 \sin\Delta_2 = 0 \\
-&\frac{dV_2}{dz}\sin\Delta_2 - \frac{d\Delta_2}{dz} V_2 \cos\Delta_2 - \frac{d\varphi}{dz} V_1 \sin\Delta_1 + C\sigma_2 V_2 \cos\Delta_2 = 0
\end{aligned} \tag{3.2}$$

Let us multiply the first equation by cos Δ_1, the second by sin Δ_1, and subtract; then the third equation by cos Δ_2, the fourth by sin Δ_2, and subtract; then the first equation by sin Δ_1, the second by cos Δ_1, and add; then the third equation by sin Δ_2, the fourth by cos Δ_2, and add. We obtain as a result

$$\begin{aligned} \frac{dV_1}{dz} &= \frac{d\varphi}{dz} V_2 \cos\Delta, & \frac{d\Delta_1}{dz} V_1 + \frac{d\varphi}{dz} V_2 \sin\Delta - C\sigma_1 V_1 &= 0 \qquad (\Delta = \Delta_1 - \Delta_2) \\ \frac{dV_2}{dz} &= -\frac{d\varphi}{dz} V_1 \cos\Delta, & \frac{d\Delta_2}{dz} V_2 + \frac{d\varphi}{dz} V_1 \sin\Delta - C\sigma_2 V_2 &= 0 \end{aligned} \qquad (3.3) \qquad (3.3)$$

Subtracting the fourth equation of the system (3.3) from the third and using the notation $V_2/V_1 = \tan\ \gamma$, we have

$$\frac{d\Delta}{dz} = C(\sigma_1 - \sigma_2) + 2\frac{d\varphi}{dz} \cot 2\gamma \sin\Delta \qquad (3.4)$$

From the first two relations (3.3) we obtain

$$\frac{d\gamma}{dz} = -\frac{d\varphi}{dz} \cos\Delta \qquad (3.5)$$

Equations (3.4) and (3.5) represent the familiar Neumann equations.

4. Let us consider the solution of equations (2.3) in some simple special cases.

1) <u>Uniformly stressed state</u> ($d\phi/dz = 0$, σ_j = const). The system (2.3) transforms to

$$\frac{dB_1'}{dz} = -iC\sigma_1 B_1', \qquad \frac{dB_2'}{dz} = -iC\sigma_2 B_2' \qquad (4.1)$$

The solution of (4.1) is

$$B_1' = B_{10}' e^{-iC\sigma_1 z}, \qquad B_2' = B_{20}' e^{-iC\sigma_2 z} \qquad (4.2)$$

where B_{10}, B_{20} are the optical electromagnetic modes at the point of entry ($z = 0$). The phase difference of the modes (4.2) is equal to $\Delta = C(\sigma_1 - \sigma_2)z$, the amplitude ratio is a constant. These results agree with the well known postulates of the photoelastic method in the solution of two-dimensional problems.

2) <u>Quasi-principal stresses varying arbitrarily in magnitude with fixed direction</u> ($d\phi/dz = 0$, $\sigma_j = \sigma_j(z)$). In this case the solution of the system (2.3) is

$$B_1' = B_{10}'e^{-ip_1}, \quad B_2' = B_{20}'e^{-ip_2} \qquad \left(p_j = C\int \sigma_j dz\right) \tag{4.3}$$

The amplitude ratio of the normal optical modes is constant, the phase /43 difference is expressed in the form

$$\Delta = C\int (\sigma_1 - \sigma_2)\, dz \tag{4.4}$$

This result is in agreement with the first approximation obtained by Proshko (ref. 9), starting with equations (1.1). The same result was obtained by Mindlin (ref. 11) for the case of linearly varying quasi-principal stresses. Equation (4.4) is the relation generally used for the analysis of frozen slices.

3) <u>Uniform rotation of quasi-principal stresses at constant magnitude</u> ($\phi = \phi_0 z$ = const, σ_j = const). The system (2.3) transforms to

$$\frac{dB_1'}{dz} = -iC\sigma_1 B_1' + \varphi_0 B_2', \qquad \frac{dB_2'}{dz} = -\varphi_0 B_1' - iC\sigma_2 B_2'. \tag{4.5}$$

Expressing the solution of the system (4.5) in the form $B_j' = B_j^* e^{-ik'z}$, we have

$$k'_{1,2} = C\left[\frac{\sigma_1 + \sigma_2}{2} \pm \frac{\sigma_1 - \sigma_2}{2}\left(1 + \frac{4\varphi_0^2}{\Delta^2}\right)^{1/2}\right], \qquad B_2^* = \frac{i(-k_1' + C\sigma_1)}{\varphi_0} B_1^* \tag{4.6}$$

For the phase difference between normal modes we obtain

$$\Delta' = k_1' - k_2' = \Delta\left(1 + \frac{4\varphi_0^2}{\Delta^2}\right)^{1/2} \tag{4.7}$$

Equations (4.6) and (4.7) lead to the same results that have been obtained by the alternative method of Drucker and Mindlin (ref. 12) and, starting with (1.1), by Ginzburg (ref. 4).

The discussion of the last two sections indicates that the general photoelasticity equations in the form (1.9) and (2.3) are equivalent to the Neumann equations and do not imply additional physical simplifications over the currently generally accepted postulates of the photoelastic method.

Application of the integral optical effect for analysis of the stressed state of three-dimensional models can in some cases be effectively substituted by the more complex freezing technique. However, the following problem arises in this connection.

In the analysis of two-dimensional photoelastic models, the customary procedure permits the determination of two experimental quantities at each point, the phase difference and isoclinic parameter. Since in the transmission of light through three-dimensional models isoclines are not generally observed in the crossed polariscope, it is supposed in the majority of investigations (see, e.g., refs. 7, 13, and 14) that only the isochrome pattern is determined experimentally. It is found in the analysis of complex-stress models, therefore, that when the number of unknowns is greater the amount of experimental data diminishes. Furthermore, it is not possible to perform a more precise measurement of the phase difference.

5. In the transmission of polarized light through photoelastic models, the components of the optical mode at the point of exit are certain functions of the components at the point of entry, the nature of this function depending on the stressed state of the model. It may be assumed that if the incident light is monochromatic and completely polarized, the light emerging from the model will be monochromatic and completely polarized. In this case, according to the work of Jones (ref. 15), the components of the optical model at the exit point will be linear functions of the components at the entry point.

Consequently, the three-dimensional photoelastic model for a given light ray will be characterized by some linear transformation or matrix, which transforms the entrant optical modes into modes at the exit point. Because the photoelastic model does not alter the light intensity, this matrix will be unitary.[1]

We denote by B_{10}, B_{20} the components of the optical mode at the entry point and by B_1, B_2 the same components at the exit point. Then the transmission of polarized light through the photoelastic model is described by the /44 transformation

$$\begin{Vmatrix} B_1 \\ B_2 \end{Vmatrix} = U \begin{Vmatrix} B_{10} \\ B_{20} \end{Vmatrix} \tag{5.1}$$

where U is a certain unitary matrix.

We note that the matrix method for describing optical systems has been thoroughly developed by R. Jones (refs. 15 and 16) for the case when the optical effects that occur must be determined in terms of the known parameters of the optical system. For solution of the converse problem, a method is still in need of development.

We will show that every unitary matrix U can be reduced to diagonal form by the transformation

$$S(\alpha_*)\, U S(-\alpha_0) = G(\varphi) \tag{5.2}$$

where

$$G(\varphi) = \begin{Vmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{Vmatrix}, \quad S(\alpha_0) = \begin{Vmatrix} \cos\alpha_0 & \sin\alpha_0 \\ -\sin\alpha_0 & \cos\alpha_0 \end{Vmatrix}, \quad S(\alpha_*) = \begin{Vmatrix} \cos\alpha_* & \sin\alpha_* \\ -\sin\alpha_* & \cos\alpha_* \end{Vmatrix} \tag{5.3}$$

---

[1] It can be shown that the matrix describing a photoelastic model is also unimodular.

The most general second-rank unitary matrix has the form

$$U = \begin{Vmatrix} e^{i\xi} \cos\theta & e^{i\zeta} \sin\theta \\ -e^{-i\zeta} \sin\theta & e^{-i\xi} \cos\theta \end{Vmatrix} \tag{5.4}$$

In the case of photoelastic models, the parameters ξ, ζ, and θ are determined by the stressed state along the light ray.

Carrying out the multiplication in equation (5.2) and equating the non-diagonal elements of the product to zero, we obtain a system of equations whose solution is

$$\mathrm{tg}\, 2\alpha_0 = \frac{\sin(\zeta + \xi)\sin 2\theta}{\sin 2\xi \cos^2\theta - \sin 2\zeta \sin^2\theta}, \quad \mathrm{tg}\, 2\alpha_* = \frac{\sin(\zeta - \xi)\sin 2\theta}{\sin 2\zeta \sin^2\theta + \sin 2\xi \cos^2\theta} \tag{5.5}$$

Since, according to equations (5.5), the quantities α_0 and α_* have real values for any values of ξ, ζ, and θ, the theorem has been proven.

Substituting α_0 and α_* from equations (5.5) into the diagonal elements of the matrix $S(\alpha_*)US(-\alpha_0)$ and equating this matrix to the matrix $G(\phi)$, we have

$$\cos 2\varphi = {}^1/_2\, [\cos 2\xi + \cos 2\zeta + \cos 2\theta\, (\cos 2\xi - \cos 2\zeta)] \tag{5.6}$$

The above results can be physically interpreted as follows. The form of the matrix U characterizing the photoelastic model depends on the choice of coordinate axes; multiplication of the matrix U by the matrix $S(-\alpha_0)$ on the right and by the matrix $S(\alpha_*)$ on the left denotes rotation of the coordinate axes at the point of entry and exit through the angles α_0 and α_*, respectively.

The directions defined by the angle α_0 are called the primary, those defined by α_* the secondary characteristic directions. The quantity 2ϕ is called the characteristic phase difference.

Denoting by $B_{10,0}$, $B_{20,0}$ the components of the entrant mode in the primary characteristic directions and by $B_{1,*}$, $B_{2,*}$ the components of the emergent mode in the secondary characteristic directions, we have

$$\begin{Vmatrix} B_{1,*} \\ B_{2,*} \end{Vmatrix} = \begin{Vmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{Vmatrix} \cdot \begin{Vmatrix} B_{10,0} \\ B_{20,0} \end{Vmatrix} \tag{5.7}$$

It is apparent from equation (5.7) that if the incident light is linearly polarized in one of the primary characteristic directions, the light emerging from the model will also be linearly polarized in the corresponding secondary characteristic direction. Mutually correspondent primary and secondary characteristic directions are said to be conjugate, the angle α between them being expressed by the relation /45

$$\mathrm{tg}\,2\alpha = \mathrm{tg}\,2(\alpha_* - \alpha_0) = \frac{2\sin 2\theta \cos\xi \cos\zeta}{\sin^2\xi - \sin^2\zeta - \cos 2\theta(\cos^2\xi + \cos^2\zeta)} \tag{5.8}$$

As evident from (5.7), the characteristic directions preserve their properties when light is passed through the model in the opposite direction, the characteristic phase difference remaining the same.

The characteristic directions can be determined experimentally by means of any polariscope in which the polarizer and analyzer are independently rotatable. The characteristic phase difference can be measured by any method of polarization optics. Since the characteristic quantities are related to the parameters of the matrix U by equations (5.5) and (5.6), while the latter in turn are governed by the stressed state of the model, determination of the characteristic parameters gives three relations between the experimental data and stress parameters. We note that if only the isochrome pattern is determined, then for each light ray only one relation will be obtained. Consequently, determination of the characteristic parameters greatly increases the amount of information on the stress state of three-dimensional models.

6. The method of characteristic directions can be applied in practice as follows: By integration of the system (1.9) or (2.3) we establish the specific form of the matrix U (5.4), which is essentially the matrix transform of the initial system of differential equations. This establishes the relations between the components of the stressed state and the parameters ξ, ζ, and Θ. The parameters ξ, ζ, and Θ are determined from the experimental characteristic parameters according to equations (5.5) and (5.6) and are used in turn to find the components of the stressed state.

In addition to the general equations (5.5) and (5.6), the relations between the characteristic parameters and stress components can also be derived directly from the solution of the original system of differential equations.

Inasmuch as the characteristic parameters depend on the wavelength,[2] the amount of information concerning the stressed state can be increased by performing experiments at different wavelengths.

We will consider, as an example based on the algorithm presented above, the problem of uniform rotation of the quasi-principal directions. It follows from solution of the original system of differential equations (4.5) that the matrix U in the given case has the form

[2]This effect is termed dispersion of the characteristic quantities.

$$U=\begin{Vmatrix} \cos\psi + iS^{-1}\sin\psi & RS^{-1}\sin\psi \\ -RS^{-1}\sin\psi & \cos\psi - iS^{-1}\sin\psi \end{Vmatrix} \tag{6.1}$$

$$(R = 2\varphi_0\Delta^{-1},\ S = \sqrt{1+R^2},\ \psi = {}^1/_2 S\Delta)$$

Equating corresponding elements of the matrices (5.4) and (6.1) we obtain the system

$$\begin{array}{ll} \cos\xi\cos\theta = \cos\psi, & \sin\xi\cos\theta = S^{-1}\sin\psi \\ \cos\zeta\sin\theta = RS^{-1}\sin\psi, & \sin\zeta\sin\theta = 0 \end{array} \tag{6.2}$$

the solution of which is

$$\operatorname{tg}\xi = S^{-1}\operatorname{tg}\psi, \quad \zeta = n\pi, \quad \sin\eta = (-1)^n RS^{-1}\sin\psi \quad (n = 0,1,2,\ldots) \tag{6.3}$$

Substituting equations (6.3) into (5.5), (5.6), and (5.8), we find

$$\begin{gathered} \operatorname{tg} 2\alpha_0 = \frac{R}{S}\operatorname{tg}\psi, \qquad \operatorname{tg} 2\alpha_* = -\frac{R}{S}\operatorname{tg}\psi, \qquad \cos 2\varphi = 1 - \frac{2}{S^2}\sin^2\psi \\ \operatorname{tg}\alpha = \operatorname{tg}(\varphi_0 + \alpha_* - \alpha_0) = \frac{\operatorname{tg}\varphi_0 - RS^{-1}\operatorname{tg}\psi}{1 + RS^{-1}\operatorname{tg}\varphi_0\operatorname{tg}\psi} \end{gathered} \tag{6.4}$$

The angle of rotation of the quasi-principal stresses and their difference (refs. 17 and 18) can be determined from the experimentally determined characteristic quantities by means of equations (6.4), which were derived by another method in reference 17.

An algorithm is given in references 1 and 2 for the case of transmission through shells.[3] We note that the method of characteristic directions should also be effective in the utilization of scattering techniques (ref. 17).

---

[3]For materials with a relatively low optical sensitivity.

The feasibility of determining three experimental variables in the trans- /46 mission of light through three-dimensional models has been demonstrated by Kuske (ref. 19) by a unique geometric method. This method is based on the application of a Poincare sphere, with the transforms of the polarized light projected on the equatorial plane of the sphere. The application of the Poincare method to the photoelasticity problem is examined in more general form by Bokshteyn (refs. 20 and 21), however the application of characteristic quantities is not treated in these papers.

In the utilization of the Poincare method, integration of the system (1.9) or (2.3) is replaced by solution of a certain spherical kinematical problem. Although the latter problem is scarcely any simpler than the former, the Poincare method deserves attention for its graphic representation of optical phenomena.

The relationship between the optical effects in photoelastic models and spherical kinematics becomes especially clear when the matrix method described above is used. It is well known that a group of unitary unimodular matrices of second rank is isomorphic with a group of orthogonal matrices of third rank. The latter group describes rotations in three-dimensional space, which can be interpreted as rotations of the Poincaré sphere. It is interesting to note in this connection that the matrix (5.4) corresponds to rotation (ref. 22) with Euler angles $\xi + \zeta$, 2θ, and $\xi - \zeta$.

REFERENCES

1. Aben, Kh. K. Application of the Photoelastic Method for the Investigation of Buckled Plates (O primenenii metoda fotouprugosti dlya issledovaniya vypuchennykh plastinok). Izvestiya Akademii Nauk Estonskogo SSR, Ser. Tekhn. i Fiz.-Mat. Nauk, Vol. 6, No. 1, 1957.
2. Aben Kh. K. and E. G. Saks. Optical Effects in Transmission through Shells (Opticheskiye yavleniya pri prosvechivanii obolochek). The Optical Polarization Method of Stress Analysis, Conference Proceedings (Polyarizatsionno-opticheskiy metod issledovaniya napryazheniy; Trudy konferentsii). Izd. LGU (Leningrad State University Publishing House), 1960.
3. Ginsburg, V. L. On the Influrence of the Terrestrial Magnetic Field on the Reflection of Radio Waves from the Ionosphere. J. Phys., Vol. 7, No. 6, 1943.
4. Ginzburg, V. L. Stress Analysis by the Optical Method (Ob issledovanii napryazheniy opticheskim metodom). Zhurnal Tekhnicheskoy Fiziki (Zh. Tekh. Fiz.), Vol. 14, No. 3, 1944.
5. Aben, Kh. K. Optical Effects in Layered Photoelastic Models (Ob opticheskikh yavleniyakh v sloistykh fotouprugikh modelyakh). Izvestiya Akademii Nauk SSSR, Otdeleniye Tekhnicheskikh Nauk (Izv. AN SSSR, OTN), Mekhanika i Mashinostroyeniye, No. 1, 1961.
6. Mindlin, R. D. and L. E. Goodman. The Optical Equations of Three-Dimensional Photoelasticity. J. Appl. Phys., Vol. 20, No. 1, 1949.
7. O'Rourke. Three-Dimensional Photoelasticity. J. Appl. Phys., Vol. 22, No. 7, 1951.

8. Proshko, V. M. Aspects of Stress Analysis in Three-Dimensional Models (Voprosy issledovaniya napryazheniy na ob"yemnykh modelyakh). Optical Polarization Method of Stress Analysis (Polyarizatsionno-opticheskiy metod issledovaniya napryazheniy). Izd. AN SSSR (Publishing House of the Academy of Sciences of the USSR), 1956.

9. Proshko, V. M. Theoretical Investigation of the Solution of a Three-Dimensional Problem by the Optical Polarization Method (Teoreticheskoye issledovaniye resheniya ob"yemnoy zadachi polyarizatsionno-optichekim metodom). Trudy Moskovskogo Instituta Inzhenernogo Zheleznodorozhnogo Transporta, No. 91, 1957.

10. Neumann, F. E. Optical Birefringence in Compressed or Nonuniformly Heated Noncrystalline Bodies (Die Gesetze der Doppelbrechung des Lichtes in komprimierten oder ungleichförmig erwärmten unkrystallinischen Körpern). Abhandlungen der koniglichen Akademien der Wissenschaften zu Berlin, 1841.

11. Mindlin, R. D. J. Appl. Mech., Vol. 8, No. 4, A-187, 1941.

12. Drucker, D. C. and R. D. Mindlin. Stress Analysis by Three-Dimensional Photoelastic Methods. J. Appl. Phys., Vol. 11, No. 11, 1940.

13. O'Rourke, R. C. and A. W. Saenz. Quenching Stresses in Transparent Isotropic Media and the Photoelastic Method. Quart. Appl. Math., Vol. 8, No. 3, 1950.

14. Drucker, D. C. and W. B. Woodward. Interpretation of Photoelastic Transmission Patterns for a Three-Dimensional Model. J. Appl. Phys., Vol. 25, No. 4, 1954.

15. Jones, R. C. A New Calculus for the Treatment of Optical Systems. V. J. Opt. Soc. Am., Vol. 37, No. 2, 1947.

16. Jones, R. C. A New Calculus for the Treatment of Optical Systems, I, II. J. Opt. Soc. Am., Vol. 31, No. 7, 1941.

17. Aben, Kh. K. Photoelastic Effects in Uniform Rotation of the Quasi-Principal Directions (Fotouprugiye yavleniya pri ravnomernom vrashchenii kvaziglavnykh napravleniy). Izv. AN SSSR, OTN, Mekhanika i Mashinostroyeniye, No. 3, 1962.

18. Aben, Kh. K. A Nomogram for Interpretation of Photoelastic Effects in Uniform Rotation of the Quasi-Principal Directions (Nomogramma dlya interpretatsii fotouprugikh yavleniy pri ravnomernom vrashchenii kvaziglavnykh napravleniy). Izv. AN SSSR, OTN, Mekhanika i Mashinostroyeniye, No. 6. 1963.

19. Kuske, A. Introduction to Stress Optics (Einführung in die Spannungsoptik). Stuttgart, 1959.

20. Bokshteyn, M. F. Geometric Analysis of the Behavior of Polorized Light in Transmission through a Three-Dimensional Stress Model (Geometricheskiy analiz povedeniya polyarizovannogo sveta pri ego prokhozhdenii cherez ob"yemnuyu napryazhennuyu model'). Optical Polarization Method of Stress Analysis. Conference Proceedings, Izd. LGU, 1960.

21. Bokshteyn, M. F. Geometric Analysis of Light Polarization in Transmission through Models Made from Optically Sensitive Materials (Geometricheskiy analiz polyarizatsii sveta pri prosvechivanii modeley iz opticheski chuvstvitel'nykh materialov). Problemy prochnosti v mashinostroyenii, No. 8, 1962.

22. Beyman, B. F. Lectures on the Application of Group Theory in Nuclear Spectroscopy (Lektsii po primeneniyu teorii grupp v yadernoy spektroskopii). Moscow, 1961.

STEMAR ENGINEERING, INC.
James S. Wood (Translator)